2.6.4 Equilibrium Equations

Since forces $\{F\}$ do not alter along the beam-column, equilibrium conditions are considered only for moments. The internal moments are obtained by substituting Eqs. (2.155) to (2.157) into Eqs. (2.143).

$$\begin{Bmatrix} N \\ M_\xi \\ M_\pi \\ M_\omega \end{Bmatrix}_{INT} = E \begin{bmatrix} A & S_y & -S_x & S_\omega \\ S_y & I_x & -I_{xy} & -I_{\omega y} \\ -S_x & -I_{xy} & I_y & I_{\omega x} \\ S_\omega & I_{\omega y} & -I_{\omega x} & -I_\omega \end{bmatrix} \begin{Bmatrix} w' & +\frac{1}{2}(u')^2 & +\frac{1}{2}(v')^2 \\ -v'' & +\theta u'' & -u'\theta' \\ u'' & +\theta v'' & -v'\theta' \\ \theta'' & -u'v''' & +v'u''' \end{Bmatrix} \tag{2.160}$$

and the twisting moment about the shear center is from Eq. (2.144),

$$\{M_\varsigma\}_{INT} = (-EI_w GK_T + \overline{K}) \begin{Bmatrix} \theta''' & -u''v''' & +v''u''' \\ \theta' & -u'v'' & +v'u'' \end{Bmatrix} \tag{2.161}$$

The external moments in the (ξ, π, ζ) coordinates are from Eqs. (2.147) and (2.148),

$$\{F_\xi\} = [R]\ \{F_{xo}\} \text{ and } \{M_\xi\} = [R]\ \{M_x\} = [R]\ (\{M_{xo}\} - [L_c]\ \{F_{xo}\}) \tag{2.162}$$

or

$$\begin{Bmatrix} F_\xi \\ F_\eta \\ F_\varsigma \end{Bmatrix}_{EXT} = \begin{bmatrix} 1 & \theta & -u' \\ -\theta & 1 & -v' \\ u' & v' & 1 \end{bmatrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix}$$

and

$$\begin{Bmatrix} M_\xi \\ M_\eta \\ M_\varsigma \end{Bmatrix}_{EXT} = \begin{bmatrix} 1 & \theta & -u' \\ -\theta & 1 & -v' \\ u' & v' & 1 \end{bmatrix} \left(\begin{Bmatrix} M_{xo} \\ M_{yo} \\ M_{zo} \end{Bmatrix} - \begin{bmatrix} 0 & -z & v \\ z & 0 & -u \\ -v & u & 0 \end{bmatrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix} \right)$$

$$= \begin{bmatrix} 1 & \theta & -u' \\ -\theta & 1 & -v' \\ u' & v' & 1 \end{bmatrix} \left(\begin{Bmatrix} M_x \\ M_y \\ M_{zo} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & -v \\ 0 & 0 & u \\ v & -u & 0 \end{bmatrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix} \right) \tag{2.162a}$$

in which M_x and M_y are bending moments of an undeflected member

$$M_x = M_{xo} + zF_{yo} \qquad M_y = M_{yo} - zF_{xo}$$

(2.163)

The external moments acting about the shear center $S(\xi_o, \eta_o)$ are from Fig. 2.20.

$$\begin{Bmatrix} \overline{M}_\xi \\ \overline{M}_\eta \\ \overline{M}_\varsigma \end{Bmatrix}_{EXT} = \begin{Bmatrix} M_\xi \\ M_\eta \\ M_\varsigma \end{Bmatrix}_{EXT} + \begin{bmatrix} 0 & 0 & -\eta_o \\ 0 & 0 & \xi_o \\ \eta_o & -\xi_o & 0 \end{bmatrix} \begin{Bmatrix} F_\xi \\ F_\eta \\ F_\varsigma \end{Bmatrix}_{EXT}$$

(2.163*a*)

Substituting Eqs. (2.162) and (2.162*a*) into Eq. (2.163*a*)

$$\begin{Bmatrix} \overline{M}_\xi \\ \overline{M}_\eta \\ \overline{M}_\varsigma \end{Bmatrix}_{EXT} = \begin{bmatrix} 1 & \theta & -u' \\ -\theta & 1 & -v' \\ u' & v' & 1 \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{zo} \end{Bmatrix}$$

$$+ \begin{bmatrix} -u'v - \eta_o u' & u'u - \eta_o v' & -v + \theta u - \eta_o \\ -vu'' + \xi_o u' & uv' + \xi_o v' & \theta v + u + \xi_o \\ v + \eta_o + \xi_o \theta & -u + \eta_o \theta - \xi_o & -u'v + v'u - \eta_o u' + \xi_o v' \end{bmatrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix}$$

(2.163*b*)

Thus, the external twisting moment about the shear center is

$$\{M_\varsigma\}_{EXT} = (u' v' 1) \begin{Bmatrix} M_x \\ M_y \\ M_{zo} \end{Bmatrix}$$

$$+ \left(v + \eta_o + \xi_o \theta - u + \eta_o \theta - \xi_o \begin{matrix} -u'v & +v'u \\ -\eta_o u' & +\xi_o v' \end{matrix}\right) \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix}$$

(2.163*c*)

Equating the internal moments in Eq. (2.160) and Eq. (2.161) to the external moments in Eq. (2.162*a*) and Eq. (2.163*c*), respectively,

$$E\begin{bmatrix} A & S_y & -S_x & -S_\omega \\ S_y & I_x & -I_{xy} & -I_{\omega y} \\ -S_x & -I_{xy} & I_y & I_{\omega x} \\ S_\omega & I_{\omega y} & -I_{\omega x} & -I_\omega \end{bmatrix} \begin{Bmatrix} w' & +\frac{1}{2}(u')^2 & +\frac{1}{2}(v')^2 \\ -v'' & +\theta u'' & -u'\theta' \\ u'' & +\theta v'' & -v'\theta' \\ \theta'' & -u'v''' & +v'u''' \end{Bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \theta & -u' & 0 \\ 0 & -\theta & 1 & -v' & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} F_{zo} & & \\ M_x & -vF_{zo} & \\ M_y & +uF_{zo} & \\ M_{zo} & +vF_{xo} & -uF_{yo} \\ M_\omega & & \end{Bmatrix}$$

(2.164)

and

$$(-EI_\omega GK_T + \overline{K}) \begin{Bmatrix} \theta''' & -u''v''' & +v''u''' \\ \theta' & -u'v'' & +v'u'' \end{Bmatrix} = (u'v'1) \begin{Bmatrix} M_x \\ M_y \\ M_{zo} \end{Bmatrix}$$

$$+(v + \eta_o + \xi_o\theta - u + \eta_o\theta - \xi_o \begin{matrix} -u'v & +v'u \\ -\eta_o u' & +\xi_o v' \end{matrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix}$$

(2.165)

Equation (2.164) is the equilibrium equation of axial force, biaxial bending moments and warping moment with respect to the origin of the arbitrary coordinates (ζ, π) on the cross section, and Eq. (2.165) is the equilibrium equation of twisting moment about the pole or torsion center (ξ_o, η_o).

Since these differential equilibrium equations are highly nonlinear, some simplifying assumptions must be adopted before an actual solution procedure is attempted.

As the first simplification, neglect the higher order terms containing the product of derivatives of displacements, then

$$E\begin{bmatrix} A & S_y & -S_x & -S_\omega \\ S_y & I_x & -I_{xy} & -I_{\omega y} \\ -S_x & -I_{xy} & I_y & I_{\omega x} \end{bmatrix} \begin{Bmatrix} w' & \\ -v'' & +\theta u'' \\ u'' & +\theta v'' \\ \theta'' & \end{Bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & -u' \\ 0 & -\theta & 1 & -v' \end{bmatrix} \begin{Bmatrix} F_{zo} & & \\ M_x & -vF_{zo} & \\ M_y & +uF_{zo} & \\ M_{zo} & +vF_{xo} & -uF_{yo} \end{Bmatrix}
$$

(2.167)

$$
(-EI_\omega GK_T + \overline{K}) \begin{Bmatrix} \theta''' \\ \theta' \end{Bmatrix} = (u'v'1) \begin{Bmatrix} M_x \\ M_y \\ M_{zo} \end{Bmatrix}
$$

$$
+ (v + \eta_o + \xi_o \theta - u + \eta_o \theta - \xi_o \begin{matrix} -u'v & +v'u \\ -\eta_o u' & +\xi_o v' \end{matrix} \begin{Bmatrix} F_{xo} \\ F_{yo} \\ F_{zo} \end{Bmatrix}
$$

(2.168)

If the principal axes are taken for the cross section coordinates (ξ, η) and the shear center $S(\xi_o, \eta_o)$ is taken as the pole of the normalized warping, Eq. (2.167) and Eq. (2.168) will be much simplified. This simplification will be presented later in Eq. (2.172).

This simplification is only applicable in elastic range. However, for a doubly symmetric section such as a wide flange shape, the shifting of the shear center from the centroid may be small even in the plastic range, thus, we may assume

$$
S_\omega = I_{\omega x} = I_{\omega y} = 0
$$

Equation of equilibrium, Eq. (2.167), is now simplified.

$$
E \begin{bmatrix} A & S_y & -S_x \\ S_y & I_x & -I_{xy} \\ -S_x & -I_{xy} & I_y \end{bmatrix} \begin{Bmatrix} w' & \\ -v'' & +\theta u'' \\ u'' & +\theta v'' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & -u' \\ 0 & -\theta & 1 & -v' \end{bmatrix} \begin{Bmatrix} F_{zo} & & \\ M_x & -vF_{zo} & \\ M_y & +uF_{zo} & \\ M_{zo} & +vF_{xo} & -uF_{yo} \end{Bmatrix}
$$

(2.169)

Now let us rewrite Eqs. (2.169) and (2.168) using the following relations [see Eq. (2.163)].

$$
F_{xo} = - M'_y
$$

$$
F_{yo} = - M'_x
$$

$$
F_{zo} = N = -P, \quad w' = \varepsilon_o
$$

(2.170)

then

$$
\begin{aligned}
&E(S_y+\theta S_x)v''+E(S_x-\theta S_y)u''-EA_{\varepsilon_o}=P\\
&-E(1_x+\theta I_{xy)})v''-E(I_{xy}-\theta I_x)u''+ES_y\varepsilon_o\\
&=\left[M_x+vP\right]+\theta\left[M_y-uP\right]-u'(M_{zo}-vM_y'-uM_x')\\
&E(I_y-\theta I_{xy})u''+E(I_{xy}+\theta I_y)v''-ES_x\varepsilon_o\\
&=\left[M_y-uP\right]-\theta\left[M_x+vP\right]-v'(M_{zo}-vM_y'-uM_x')\\
&-EI_w\theta'''+(GK_T+\overline{K})\theta'=u'\left[M_x+(v+\pi_o)P\right]\\
&+v'\left[M_y-(u+\xi_o)P\right]+M_{zo}-(v+\xi_y\theta+\eta_o\theta+\xi_o)M_x'
\end{aligned}
\tag{2.171}
$$

These are the basic equations to be used in Chap. 7 for plastic analysis of beam-columns [see Eq. (7.1)] where the sectional properties S_x, I_x etc., are evaluated for the elastic, unyielded portion of the cross section.

In an elastic analysis, if ξ and η are the principal axes of the cross section and ω is the normalized unit warping with respect to the shear center $S(\xi_o, \eta_o)$,

$$
S_x=S_y=S_\omega=I_{xy}=I_{\omega x}=I_{\omega y}=0
$$

then, Eqs. (2.171) are reduced to

$$
\begin{aligned}
&-EA\varepsilon_o=P\\
&-EI_x(v''-\theta u'')=(v-\theta u)P+M_x+\theta M_y-u'(M_{zo}-vM_y'-uM_x')\\
&EI_y(u''+\theta v'')=-(u+\theta v)P+M_y-\theta M_x-v'(M_{zo}-vM_y'-uM_x')\\
&-EI_\omega\theta'''+(GK_T+\overline{K})\theta'=u'\left[M_x+(v+\pi_o)P\right]+v'\left[M_y-(u+\xi_o)P\right]\\
&+M_{zo}-(v+\xi_o\theta+\eta_o)M_y'-(u-\eta_o\theta+\xi_o)M_x'
\end{aligned}
\tag{2.172}
$$

If we neglect all nonlinear terms of displacements, the last three equations of Eq. (2.172) are reduced to

$$
\begin{aligned}
&EI_xv''+vP+M_x+\theta M_y-u'M_{zo}=0\\
&EI_yu''+uP-M_y+\theta M_x+v'M_{zo}=0\\
&EI_\omega\theta'''-(GK_T+\overline{K})\theta'\\
&+u'(M_x+\eta_oP)+v'(M_y-oP)+M_{zo}\\
&-(v+\xi_o\theta+\eta_o)M_y'-(u-\eta_o\theta+\xi_o)M_x'=0
\end{aligned}
\tag{2.172a}
$$

In the case of elastic section, the shear center $S(\xi_o, \eta_o)$ can be clearly defined. Further, the rotation of the section θ takes place with respect to the shear center; thus Eq. (2.172) can be rewritten in terms of displacements of the shear center,

$$u_o = u - \theta\eta_o \quad \text{and} \qquad v_o = v + \theta\xi_o$$

Thus, Eq. (2.172a) becomes

$$\begin{aligned}
&EI_x(v_o'' - \theta''\xi_o) + (v_o - \theta\xi_o)P + M_x + \theta M_y - (u_o' + \theta'\eta_o)M_{zo} = 0 \\
&EI_y(u_o'' + \theta''\eta_o) + (u_o + \theta\eta_o)P - M_y + \theta M_x + (v_o' - \theta'\xi_o)M_{zo} = 0 \\
&EI_\omega\theta''' - (GK_T + \overline{K}^*)\theta' + (u_o' - v_o'\xi_o)P + (\xi_o^2 + \eta_o^2)P\theta' \\
&+ (u_o' + \theta'\eta_o)M_x + (v_o' - \theta'\xi_o)M_y + M_{zo} - (v_o + \eta_o)M_y' - (u_o + \xi_o)M_x' = 0
\end{aligned} \tag{2.173}$$

These are the exact equations of linear theory. The first two are equilibrium equations of bending moments about the centroidal axes ξ and η. The last one is the equilibrium equation of twisting moment about the shear center, $S(\xi_o, \eta_o)$. The variables, u_o, v_o and θ are displacements and rotation about the shear center.

For simplicity, in most stability analysis, θ is assumed to be small compared with u_o and v_o, and the terms $\theta'\xi_o$, $\theta'\eta_o$, $\theta''\xi_o$, $\theta''\eta_o$, are often neglected except the terms related axial force P which is the most significant contribution to stability. In this case, Eqs. (2.173) are much simplified as

$$\begin{aligned}
&EI_x v_o'' + (v_o - \theta\xi_o)P + M_x + \theta M_y - u_o' M_{zo} = 0 \\
&EI_y u_o'' + (u_o + \theta\eta_o)P - M_y + \theta M_x + v_o' M_{zo} = 0 \\
&EI_\omega\theta''' - (GK_T + \overline{K}^*)\theta' + (u_o'\eta_o - v_o'\xi_o)P \\
&+ u_o' M_x + v_o' M_y + M_{zo} - (v_o + \eta_o)M_y' - (u_o + \xi_o)M_x' = 0
\end{aligned} \tag{2.174}$$

in which

$$\overline{K}^* = \overline{K} - (\xi_o^2 + \eta_o^2)P = \int \sigma\left[(\xi - \xi_o)^2 + (\eta - \eta_o)^2\right]dA - (\xi_o^2 + \eta_o^2)P \tag{2.174a}$$

is the modified Wagner coefficient. In the case of uniformly distributed axial stress,

$\sigma = \dfrac{P}{A}$. Then,

$$\overline{K}^* = \frac{P}{A}(I_x + I_y) \tag{2.174b}$$

Hereinafter, this modified Wagner coefficient is expressed simply as $\overline{K}$.

In the case that the cross section is doubly symmetric so that the shear center considers with the centroid, we have

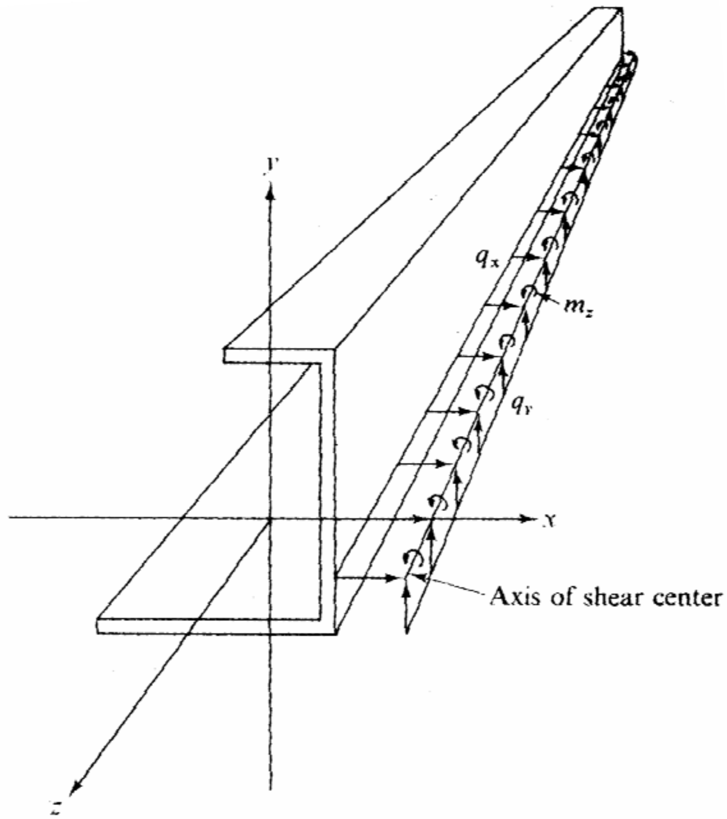

FIGURE 2.21
Beam-column with distributed load

$$\xi_o = \eta_o = 0 \tag{2.175}$$

then Eqs. (2.174) can be further reduced to

$$\begin{aligned} &EI_x v'' + vP + M_x + \theta M_y - u'M_{zo} = 0 \\ &EI_y u'' + uP - M_y + \theta M_x + v'M_{zo} = 0 \\ &EI_\omega \theta''' - (GK_T + \overline{K})\theta' + u'M_x + v'M_y + M_{zo} - vM'_y - uM'_x = 0 \end{aligned} \tag{2.176}$$

Further, if the load is applied symmetrically at both ends,

$$M_x = M_{xo},\quad M_y = M_{yo},\quad M_{zo} = 0,\quad M'_x = M'_y = 0 \tag{2.177}$$

thus

$$\begin{aligned} &EI_x v'' + vP + M_{xo} + \theta M_{yo} = 0 \\ &EI_y u'' + uP - M_{yo} + \theta M_{xo} = 0 \\ &EI_\omega \theta''' - (GK_T + \overline{K})\theta' + u'M_{xo} + v'M_{yo} = 0 \end{aligned} \tag{2.178}$$

These are the most fundamental differential equations of beam-columns in space.

If a set of uniformly distributed loads q_x, q_y and m_z are applied along the shear center as shown in Fig. 2.21, the following fourth order differential equations instead of the simplified Eqs. (2.174) are to be used, i.e.,

$$\begin{aligned} &EI_x v_o^{IV} + (v_o'' - \theta''\xi_o)P + \theta''M_y + 2\theta'M_y' + \theta M_y'' - u_o'''M_{zo} - 2u_o''M_{zo}' + M_x'' = 0 \\ &EI_y u_o^{IV} + (u_o'' + \theta''\pi_o)P + \theta''M_x + 2\theta'M_x' + \theta M_x'' + v_0'''M_{zo} + 2v_0''M_{zo}' - M_y'' = 0 \\ &EI_\omega \theta^{IV} - (GK_T + \overline{K})\theta'' - \overline{K}'\theta' + u_o''(\pi_o P + M_x) - v_o''(\xi_o P - M_y) \\ &-(v_o + \eta_o)M_y'' - (u_o + \xi_o)M_x'' + M_z' = 0 \end{aligned} \tag{2.179a}$$

in which

$$\begin{aligned} &M_x = M_{xo} + F_{yo}z - \frac{1}{2}q_y z^2 \\ &M_y = M_{yo} - F_{xo}z + \frac{1}{2}q_x z^2 \\ &M_z = M_{zo} - M_z z \end{aligned} \tag{2.179b}$$

are to be substituted instead of Eq. (2.163). Equations (2.179*a*) are the general differential equations used in Chap. 4. Boundary conditions for these differential equations are

displacement	u, v (= 0 if supported)
slope	u', v' (= 0 if fixed)
bending moment	$EI_x v'' = M_{xo} + \theta M_{yo}$ (= 0 if simply supported)
	$EI_y u'' = M_{yo} + \theta M_{xo}$
shear force	$EI_x v''' = - V_y - \theta V_x$ (= 0 for free end)
	$EI_y' u''' = - V_x + \theta V_y$

rotation angle	θ (= 0 if restrained)
twist	θ' (= 0 if warping is restrained)
warping moment	$EI_\omega\theta'' = M_\omega$ (= 0 if warping free)
twisting moment	$EI_\omega\theta''' - (GK_T + \overline{K})\,\theta' = M_z$ (= 0 for free end)

(2.180)

2.7 Summary

In the analysis of torsion, the coordinate of double sectorial area or unit warping plays an important role (Fig. 2.13)

centroidal $\omega = \int_o^\varsigma \rho ds$

shear center $\omega_o = \int_o^\varsigma \rho_o ds = \omega + \eta_o(\xi - \xi_A) - \xi_o(\eta - \eta_A)$

normalized $\omega_n = \frac{1}{A}\int_o^E \omega_o t ds - \omega_o$

(2.181)

Equations have highly nonlinear terms as was seen in Eq. (2.164) and Eq. (2.165). These equations are derived using the rotation matrix, Eq. (2.154) instead of Eq. (2.153). If all the nonlinear terms of displacements and small θ terms are neglected, we have the approximate equations

$$
\begin{aligned}
&EI_x v_o'' + (v_o - \theta\xi_o)P + M_x + \theta M_y - u_o' M_{zo} = 0 \\
&EI_y u_o'' + (u_o + \theta\eta_o)P - M_y + \theta M_x + v_o' M_{zo} = 0 \\
&EI_\omega\theta''' - (GK_T + \overline{K})\theta' + (u_o'\eta_o - v_o'\xi_o)P + u_o' M_x + v_o' M_y + M_{zo} \\
&-(v_o + \eta_o)M_y' - (u_o + \xi_o)M_x' = 0
\end{aligned}
$$

(2.166)

These simplified equations are often used as the basis for analyses. It is to be noted that the first two equations are equilibrium of bending moment about the principal axes on the cross section while the last equation is equilibrium of twisting moment about the shear center $S(\xi_o, \eta_o)$. All equations are written in terms of displacements of the shear center: u_o, v_o and θ.